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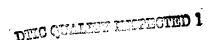Stennis Space Center, MS 39529-5004

Formulation of the NRL Layered Ocean Model in Spherical Coordinates

DANIEL R. MOORE
ALAN J. WALLCRAFT

Planning Systems Incorporated
Slidell, LA

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*Planning Systems Incorporated, 115 Christian Lane, Slidell, LA 70458

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This report describes NRL's hydrodynamic (isopycnal) nonlinear, primitive equation, ocean circulation model in a spherical layer suitable for a large basin or the global domain. This document should be read in conjunction with NOARL Report 35, December 1991, The Navy Layered Ocean Model User's Guide [25].

Spherical geometry requires careful treatment of the equations of fluid dynamics to ensure spurious terms are not introduced during the layer averaging process, particularly in the momentum diffusion expressions.

From these layer averaged equations, finite difference approximations are derived on a grid of uniform intervals in longitude and latitude (not necessarily the same). These equations may be solved efficiently on modern architecture scientific computers, allowing accurate description of ocean flows.

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CONTENTS

EXECUTIVE SUMMARY

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Spherical geometry requires careful treatment of the equations of fluid dynamics to ensure spurious terms are not introduced during the layer averaging process, particularly in the momentum diffusion expressions.

From these layer average equations, finite difference approximations are derived on a grid of uniform intervals in longitude and latitude (not necessarily the same). The resulting finite difference equations may be solved efficiently on modern architecture scientific computers to describe accurately ocean currents, temperature and salinity distributions over large time and space scales.

Formulation of the NRL Layered Ocean Model in Spherical Coordinates

1. PRELIMINARIES

1.1 Motivation

Ocean Modeling has evolved in 20 years from simple limited resolution, limited area, ideal domain, Cartesian geometry computations [7], [21], [8] to realistic basin scale eddy resolving ocean models [6], [10], [9]. This progress has reflected both the growth in computing capacity and in the sophistication and experience of numerical oceanographers. However, the actual models cannot form a continuous range as the transition from Cartesian to spherical geometry results in substantial changes in the appearance of the vector calculus operators and of the equations of fluid flow, in spite of the coordinate invariance of the underlying physics and mathematics. This change in appearance reflects the difficulties of expressing the laws of physics in a curvilinear coordinate system, particularly the conservation of vector quantities such as momentum.

Another difficulty is that while the choice and orientation of {x,y,z} to label the orthogonal Cartesian coordinates appears to be common across most physical disciplines, no such consensus exists for the choice and orientation of spherical or spheroidal coordinates. While most mathematical and theoretical studies favour co-latitude as the declination coordinate, many oceanographers and meteorologist prefer latitude. However, introductory mathematics and physics text books tend to opt for the former choice. This can lead to confusion and error. The labels assigned to the three spherical surface coordinates also vary widely, even within the meteorology and oceanography literature.

For simplicity, the criteria of minimizing differences between the Cartesian and spherical models will guide the choices of notation and orientation used in this presentation.

A note of caution. Care will need to be taken to distinguish two-dimensional mathematical operations defined on the surface of a sphere and three-dimensional physical and mathematical processes defined within a thin spherical layer. The purpose of the *Shallow Water* set of approximations is to reduce the equations that apply to three-dimensional physical flow throughout a thin fluid layer to equations describing a two-dimensional model flow on a spherical surface in a way that preserves as much of the physics of the 3-D flow as possible.

1.2 Coordinates and unit vectors.

To conform to the conventional right-handed (x, y, z) Cartesian coordinate system locally, we chose longitude, latitude and radius as our surface spherical coordinate directions *in that order*.

We label the longitude and latitude angles ϕ and θ respectively. We assume that ϕ increases in an eastward direction, and that θ increases in a northward direction from a value of zero at the equator, taking negative values in the Southern hemisphere. The two spherical surface coordinate singularities (the North and South Poles) are assumed to coincide with the rotation axis of the earth, unless stated otherwise. The origin of the longitude (ϕ) coordinate can be chosen for convenience, with the Greenwich meridian assumed to be the default. The surface of the earth is chosen to lie on a sphere at a constant radius a. The radial coordinate is chosen to increase outwards. We will label this coordinate r. As the angles ϕ and θ are dimensionless, physical distances will be given implicitly in the form $a\,\phi$ or $a\,\theta$, where a is a suitable mean reference radius for the earth.

We label the unit vectors that are locally tangent to the surface and parallel to lines of constant latitude and constant longitude as $\vec{e}_\phi$ and $\vec{e}_\theta$ respectively. We assume that $\vec{e}_\phi$ points eastward and $\vec{e}_\theta$ points northward. There is a local unit normal vector $\vec{e}_r$ pointing out of the sphere at each point (ϕ, θ, a) on the spherical surface.

The distinction between a sphere and true geopotential surfaces will be ignored. Gill [4] pp 91 - 92 contains a full discussion of the error incurred in approximating the shape of the earth by a sphere. There he estimates that a maximum error of 0.17% occurs in the geometrical factors in the various equations if the earth's figure is approximated as a sphere instead of an oblate spheroid. Thus, although in reality horizontal fluid flows are locally parallel to geopotential surfaces, very little error is made approximating these complicated surfaces by concentric spheres.

1.3 History

The derivation of the spherical *Shallow Water* equations from the full 3-D Navier Stokes equations in a spherical coordinate system has been carried out by several authors in the past [11], [17]. While early agreement was reached as to the proper form for the momentum advection and conservation of mass expressions, opinions continue to differ as to the correct form for the layer averaged viscous dissipation in the presence of layer thickness and horizontal density variations. We followed Wajsowicz [24] and Shchepetkin & O'Brien [18] in the derivation of these terms and can demonstrate that our forms generate no spurious forces for analytic test cases such as rigid rotation of a varying thickness shell.

Given the complexity of the appearance of these equations in spherical coordinates, many mathematically equivalent expressions are possible. These variations in appearance can lead to variations in the finite difference approximations used when replacing these partial differential equations by difference equations. These can give rise to widely differing *answers* to the same underlying mathematical problem. We note that while the equations given in Section 2 are mathematically equivalent to those appearing in Shchepetkin and O'Brien (1996) [18], the most recently published contribution to this discussion that we are aware of; they are not identical. They differ in the form taken for the viscous dissipation, although they adhere to the requirement that this term be the divergence of a symmetric tensor. The form we use is taken from Wajsowicz (1994) [24].

2. SPHERICAL SHELL FLUID DYNAMICS

The reduction of the three dimensional partial differential equations describing the conservation of mass, momentum, energy and matter *(tracers)* at each point in an ocean layer to a set of two-dimensional partial differential equations for flow on a spherical layer is done by ntegrating these equations in the radial direction over the thickness of the spherical shell, h from $r = a$ to $r = a + h$. This averaging process requires application of boundary conditions at the top and bottom of each layer and the conversion of surface stresses into body forces. In our equations these are subsumed into the "sources − sinks " terms.

In Cartesian coordinates, the integration is straightforward and yields the equations set out in *NOARL35* [25] or Hurlburt and Thompson [8]. On a spherical shell the averaging must be done with care to avoid generating spurious sources or sinks of energy, mass or momentum from curvature terms. After a lively discussion in the literature (sf. [1], [2], [5], [11], [12], [13], [15] [17] [20] [23] & [26]) a consensus has emerged as to the correct forms for these equations and we have chosen a representation that closely resembles the one used in our earlier Cartesian work [25].

2.1 2-D Spherical Shell Velocity Field

Classically, the derivation of the *Shallow Water* equations assumes each layer is homogeneous in the vertical direction and that vertical velocities within the layer can be ignored except to balance horizontal flow divergence and conserve mass. While this vertical uniformity assumption is appropriate in Cartesian coordinates, it introduces spurious shear forces in a spherical layer. It is necessary to assume that the horizontal velocity has a radial dependence. This ensures that particles at the top and bottom of the layer that lie on a line through the center of the sphere have the same angular velocity about the center of the sphere and hence remain *in step* for any horizontal velocity field.

The appropriate form for an incompressible velocity field ($\vec{u}$) in a thin spherical layer is:

$$\vec{u}(\phi, \theta, r, t) = \left(\left(\frac{r}{a}\right) u'(\phi, \theta, t), \ \left(\frac{r}{a}\right) v'(\phi, \theta, t), \ \frac{A(\phi, \theta, t)r^3 + B(\phi, \theta, t)}{r^2} \right), \tag{1}$$

where A & B are chosen to match the conditions:

$$w(\phi, \theta, r = a, t) = w_b(\phi, \theta, t), \tag{2a}$$

$$w(\phi, \theta, r = a + h, t) = w_b(\phi, \theta, t) + \frac{dh}{dt}. \tag{2b}$$

This linear radial dependence of the horizontal components of the velocity field is necessary to keep the top of a column of fluid positioned over its bottom for all horizontal flows. The entire layer moves coherently as it flows over the surface of the sphere and there is no shear between fluid at different radii. The vertical velocity is chosen to match the vertical motion of the lower boundary and the local divergence of the horizontal motion field. This maintains the incompressibility of the overall flow.

For reference:

$$w(\phi, \theta, r, t) = \frac{(r^3 - a^3)(a + h)^2 \frac{dh}{dt} + [(r^3 + ha^2)(a + h)^2 - r^3 a^2] w_b}{r^2 [(a + h)^3 - a^3]}. \tag{3}$$

In the limit $\left(\frac{h}{a}\right) \ll 1$:

$$\frac{Ar^3 + B}{r^2} \to w_b(\phi, \theta, t) + \left(\frac{r - a}{h}\right) \frac{dh}{dt}, \tag{4}$$

similar to the Cartesian result [25].

The leading error term is :

$$-\left(\frac{r - a}{h}\right)\left(\frac{a + h - r}{a}\right)\left[\frac{dh}{dt} + \left(\frac{h}{a}\right) w_b\right]. \tag{5}$$

This vanishes at $r = a$ and $r = a + h$ and is bounded in that interval by:

$$-\frac{1}{4}\left(\frac{h}{a}\right)\left[\frac{dh}{dt} + \left(\frac{h}{a}\right) w_b\right]. \tag{6}$$

The divergence of this velocity field (1)is:

$$\frac{1}{a\cos\theta}\left[\frac{\partial u'}{\partial \phi} + \frac{\partial(v'\cos\theta)}{\partial \theta}\right] + \frac{3(a + h)^2}{(a + h)^3 - a^3}\frac{dh}{dt} + \frac{3\left[(a + h)^2 - a^2\right]}{(a + h)^3 - a^3}w_b. \tag{7}$$

As this relationship does not depend upon the vertical coordinate, it is valid everywhere in the fluid layer.

In the limit $\left(\frac{h}{a}\right) \to 0$, we recover *(for an incompressible fluid)* a two-dimensional continuity equation :

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\theta}\left(\frac{\partial(hu')}{\partial \phi} + \frac{\partial(hv'\cos\theta)}{\partial \theta}\right) \approx 0 \tag{8}$$

or

$$\frac{\partial h}{\partial t} + \nabla_2 \cdot \vec{V} \approx 0, \tag{9}$$

(using the subscript $_2$ to denote spherical surface two dimensional vector calculus operators) where $\vec{V}$ is the horizontal velocity transport vector *(see next page)*.

The leading error term is

$$\left(\frac{h}{3a}\right)\left[2w_b - \left(\frac{h}{a}\right)\left(\frac{dh}{dt} + w_b\right)\right], \tag{10}$$

when approximating (7) by (9). In the absence of vertical motion of the lower boundary, (9) is accurate to $O\left(\frac{h}{a}\right)^2$.

2.2 A Single Layer with Uniform Density

Here are the equations used within a single layer. If this is one of a stack of layers, interactions between layers are mediated by the "sources - sinks" terms in each equation. The effective pressure terms $\frac{\partial h}{\partial \phi}$, $\frac{\partial h}{\partial \theta}$ will have to be modified as in *NOARL35* [25] to reflect the possibilties of barotropic and baroclinic modes of motion. Hereafter we will drop the prime to distinguish 2-D velocities from 3-D velocities and assume that (u, v) label horizontal layer averaged velocity components.

2.2.1 Continuity

$$\frac{\partial h}{\partial t} + \frac{1}{a \cos \theta} \left[\frac{\partial U}{\partial \phi} + \frac{\partial (V \cos \theta)}{\partial \theta} \right] = \text{sources - sinks.} \tag{11}$$

2.2.2 Velocity Transport

Define an angular deformation tensor $\vec{\vec{e}}$ to have components:

$$e_{\phi\phi} = \frac{\partial}{\partial \phi} \left(\frac{u}{\cos \theta} \right) - \cos \theta \frac{\partial}{\partial \theta} \left(\frac{v}{\cos \theta} \right), \quad (= -e_{\theta\theta}) \tag{12a}$$

$$e_{\phi\theta} = \frac{\partial}{\partial \phi} \left(\frac{v}{\cos \theta} \right) + \cos \theta \frac{\partial}{\partial \theta} \left(\frac{u}{\cos \theta} \right). \quad (= e_{\theta\phi}) \tag{12b}$$

where the velocities (u, v) are defined from the transports (U, V) by:

$$h\,u = U, \tag{13a}$$

$$h\,v = V. \tag{13b}$$

Then the two components of the Velocity Transport equation may be written:

$\vec{e}_\phi$ **component:**

$$\frac{\partial U}{\partial t} + \frac{1}{a \cos \theta} \left[\frac{\partial (U\,u)}{\partial \phi} + \frac{\partial (V\,u \cos \theta)}{\partial \theta} - V\,u \sin \theta - a\,\Omega\,V \sin 2\theta \right] = -\frac{h\,g}{a \cos \theta} \frac{\partial h}{\partial \phi}$$

$$+ hF_\phi + \frac{A_H}{a^2 \cos^2 \theta} \left[\frac{\partial (h \cos \theta\, e_{\phi\phi})}{\partial \phi} + \frac{\partial (h \cos^2 \theta\, e_{\phi\theta})}{\partial \theta} \right] + \text{sources - sinks,} \tag{14a}$$

$\vec{e}_\theta$ **component:**

$$\frac{\partial V}{\partial t} + \frac{1}{a \cos \theta} \left[\frac{\partial (U\,v)}{\partial \phi} + \frac{\partial (V\,v \cos \theta)}{\partial \theta} + U\,u \sin \theta + a\,\Omega\,U \sin 2\theta \right] = -\frac{h\,g}{a} \frac{\partial h}{\partial \theta}$$

$$+ hF_\theta + \frac{A_H}{a^2 \cos^2 \theta} \left[\frac{\partial (h \cos \theta\, e_{\phi\theta})}{\partial \phi} - \frac{\partial (h \cos^2 \theta\, e_{\phi\phi})}{\partial \theta} \right] + \text{sources - sinks.} \tag{14b}$$

2.2.3 Scalar transport

$$\frac{\partial(h\,T)}{\partial t} + \frac{1}{a\,\cos\theta}\left[\frac{\partial(U\,T)}{\partial\phi} + \frac{\partial(V\,T\,\cos\theta)}{\partial\theta}\right] =$$

$$+\frac{\kappa}{a^2\cos^2\theta}\left[\frac{\partial}{\partial\phi}\left(h\,\frac{\partial T}{\partial\phi}\right) + \cos\theta\,\frac{\partial}{\partial\theta}\left(h\cos\theta\,\frac{\partial T}{\partial\theta}\right)\right] + \text{sources - sinks.} \qquad (15)$$

Here h is the vertical layer thickness; Ω is the rotation rate of the earth; g is the acceleration due to gravity; F_ϕ and F_θ are the components of the applied forcing fields in local (ϕ, θ) components; A_H is the eddy viscosity; T is a scalar tracer; and κ is the coefficient of diffusivity for the scalar T.

2.3 Discussion

These forms are mathematically equivalent to those used in Shchepetkin and O'Brien (1996) [18]. The form of the diffusion operator is taken from Wajsowicz (1994) [24], assuming that the vertical diffusivity and second viscosity coefficients vanish. This form of the friction term should be more accurate for layers with varying thickness than the form used in some earlier Cartesian studies: $A_H\nabla^2\vec{U}$, in Hurlburt and Thompson (1980) [8], although it will require more work to calculate.

Equations (12a-b) imply a stress-strain relationship of the form

$$\vec{\sigma} = A_H\,h\,\cos\theta\,\vec{e}, \qquad (16)$$

and a stress force contribution to the transport equations of the form:

$$\frac{1}{a\,\cos\theta}\,\nabla_2\cdot\vec{\sigma},$$

where $\nabla_2\cdot$ is defined in Appendix B (B5). $\vec{\sigma}$ and $\vec{e}$ are angular tensors. These require a factor of $\frac{1}{a}$ to transform them to physical tensors [22]. The components of $\vec{e}$ may be related to the rate-of-strain tensor $\vec{\mathcal{E}}$ components (as defined in [3]) for the velocity field (1):

$$e_{\phi\phi} = a\,(\mathcal{E}_{\phi\phi} - \mathcal{E}_{\theta\theta}), \qquad e_{\phi\theta} = 2\,a\,\mathcal{E}_{\phi\theta}. \qquad (17)$$

Equations (11) - (15) are valid only in the limits $\left(\frac{h}{a}\right) \ll 1$ and $u, v \gg w$. Typical terms ignored when deriving these simplified equations are of the order of $\left(\frac{h}{a}\right)^2$.

2.4 Enstrophy Conserving Formulation

An alternative formulation (as recommended by [17], [1]) uses the radial vorticity component ζ defined as :

$$\zeta = \frac{1}{a\cos\theta}\left\{ \frac{\partial v}{\partial \phi} - \frac{\partial(u\cos\theta)}{\partial \theta} \right\}, \tag{18}$$

and exploits the vector identity

$$(\vec{u}\cdot\nabla)\,\vec{u} = (\nabla\times\vec{u})\times\vec{u} + \nabla\left(\frac{\vec{u}\cdot\vec{u}}{2}\right). \tag{19}$$

This substitution yields:

$$\frac{\partial U}{\partial t} + \frac{1}{a\cos\theta}\left[2u\frac{\partial U}{\partial \phi} + v\frac{\partial V}{\partial \phi} + u\frac{\partial(V\cos\theta)}{\partial \theta} \right] - (\zeta + 2\Omega\sin\theta)\,V, \tag{20a}$$

$$\frac{\partial V}{\partial t} + \frac{1}{a}\left[2v\frac{\partial V}{\partial \theta} + u\frac{\partial U}{\partial \theta} + \frac{v}{\cos\theta} - \frac{\tan\theta V v}{a} \right] + (\zeta + 2\Omega\sin\theta)\,U, \tag{20b}$$

as an alternative left hand side for equations (14a) and (14b).

If $\vec{u}$ is used instead of $\vec{V}$ as the prognostic variable, then the left hand sides of these equations assume the simpler forms [1]:

$$h\frac{\partial u}{\partial t} + \frac{h}{a\cos\theta}\frac{\partial}{\partial \phi}\left(\frac{u^2+v^2}{2}\right) - (2\Omega\sin\theta + \zeta)\,V \quad = \quad \ldots, \tag{21a}$$

$$h\frac{\partial v}{\partial t} + \frac{h}{a}\frac{\partial}{\partial \theta}\left(\frac{u^2+v^2}{2}\right) + (2\Omega\sin\theta + \zeta)\,U \quad = \quad \ldots. \tag{21b}$$

2.5 Relocating the Poles

Converging coordinate lines may cause difficulties with many numerical schemes, particularly in limiting the size of the permissible time steps. A ploy to lessen this problem is to move the coordinate singularities so that they both lie over land [19]. If this is done, the Coriolis term in the momentum transport equations needs to reflect this loss of coincidence of the rotation axis and the North and South Poles of the coordinate system.

In vector form the Coriolis pseudo-force in the velocity transport equations (14a,14b) is expressed as $2\,\vec{\Omega} \times \vec{U}$. Using

$$\vec{\Omega} = \left(\cos\phi_\Omega \, \sin\theta_\Omega \;\; \vec{\mathbf{i}} + \sin\phi_\Omega \, \sin\theta_\Omega \;\; \vec{\mathbf{j}} + \sin\theta_\Omega \;\; \vec{\mathbf{k}} \right)$$

and re-expressing $\vec{\Omega}$ in the local spherical basis *(using equations (A1a-c) from Appendix A)* we find:

$$
\begin{aligned}
\vec{\Omega} = \quad & \Omega \cos\theta_\Omega \sin(\phi_\Omega - \phi) && \vec{\mathbf{e}}_\phi \\
+ \; & \Omega \left(\sin\theta_\Omega \cos\phi - \cos\theta_\Omega \sin\theta \cos(\phi_\Omega - \phi) \right) \vec{\mathbf{e}}_\theta \\
+ \; & \Omega \left(\sin\theta_\Omega \sin\theta + \cos\theta_\Omega \cos\theta \cos(\phi_\Omega - \phi) \right) \vec{\mathbf{e}}_\mathbf{r}, \\
\\
= \; & (\Omega_\phi, \; \Omega_\theta, \; \Omega_r).
\end{aligned}
$$

For $\vec{U} = (U, V, 0)$ the Coriolis pseudo-force vector becomes $2\,(-V\,\Omega_r,\; U\,\Omega_r,\; V\,\Omega_\phi - U\,\Omega_\theta)$ or, in spherical surface coordinates, $(-V\,f,\; U\,f)$ where:

$$f = 2\,\Omega \left(\sin\theta_\Omega \sin\theta + \cos\theta_\Omega \cos\theta \cos(\phi_\Omega - \phi) \right). \tag{22}$$

Thus, on a shifted coordinate spherical surface, the Coriolis term becomes dependent on both the local longitude and latitude, but retains its $2\,\Omega\,(-V\,f,\; U\,f)$ form.

3. FINITE DIFFERENCE APPROXIMATIONS TO THE EQUATIONS

3.1 The grid on a sphere

Finite difference approximations to the equations set out in Section 2 are defined on a uniform $(\Delta\phi, \Delta\theta)$ grid covering most of the earth, but commonly excluding extreme latitudes. The 'C' grid of [13] defines the spatial staggered grids where the h, U and V variables take discrete values. Figure 1 sets out the indexing of these variables on the three interlocked staggered meshes.

θ	ϕ_1	$\phi_{\frac{3}{2}}$	ϕ_2	$\phi_{\frac{5}{2}}$	ϕ_3	$\phi_{\frac{7}{2}}$	ϕ_4	$\phi_{\frac{9}{2}}$	ϕ_5	$\phi_{\frac{11}{2}}$	ϕ_6	$\phi_{\frac{13}{2}}$	ϕ_7	$\phi_{\frac{15}{2}}$
$\theta_{\frac{15}{2}}$	$V_{7,1}$	|	$V_{7,2}$		$V_{7,3}$		$V_{7,4}$		$V_{7,5}$		$V_{7,6}$		$V_{7,7}$	
θ_7	$h_{7,1}$	$U_{7,1}$	$h_{7,2}$	$U_{7,2}$	$h_{7,3}$	$U_{7,3}$	$h_{7,4}$	$U_{7,4}$	$h_{7,5}$	$U_{7,5}$	$h_{7,6}$	$U_{7,6}$	$h_{7,7}$	$U_{7,7}$
$\theta_{\frac{13}{2}}$	$V_{6,1}$	|	$V_{6,2}$		$V_{6,3}$		$V_{6,4}$		$V_{6,5}$		$V_{6,6}$		$V_{6,7}$	
θ_6	$h_{6,1}$	$U_{6,1}$	$h_{6,2}$	$U_{6,2}$	$h_{6,3}$	$U_{6,3}$	$h_{6,4}$	$U_{6,4}$	$h_{6,5}$	$U_{6,5}$	$h_{6,6}$	$U_{6,6}$	$h_{6,7}$	$U_{6,7}$
$\theta_{\frac{11}{2}}$	$V_{5,1}$	|	$V_{5,2}$		$V_{5,3}$		$V_{5,4}$		$V_{5,5}$		$V_{5,6}$		$V_{5,7}$	
θ_5	$h_{5,1}$	$U_{5,1}$	$h_{5,2}$	$U_{5,2}$	$h_{5,3}$	$U_{5,3}$	$h_{5,4}$	$U_{5,4}$	$h_{5,5}$	$U_{5,5}$	$h_{5,6}$	$U_{5,6}$	$h_{5,7}$	$U_{5,7}$
$\theta_{\frac{9}{2}}$	$V_{4,1}$	|	$V_{4,2}$		$V_{4,3}$		$V_{4,4}$		$V_{4,5}$		$V_{4,6}$		$V_{4,7}$	
θ_4	$h_{4,1}$	$U_{4,1}$	$h_{4,2}$	$U_{4,2}$	$h_{4,3}$	$U_{4,3}$	$h_{4,4}$	$U_{4,4}$	$h_{4,5}$	$U_{4,5}$	$h_{4,6}$	$U_{4,6}$	$h_{4,7}$	$U_{4,7}$
$\theta_{\frac{7}{2}}$	$V_{3,1}$	|	$V_{3,2}$		$V_{3,3}$		$V_{3,4}$		$V_{3,5}$		$V_{3,6}$		$V_{3,7}$	
θ_3	$h_{3,1}$	$U_{3,1}$	$h_{3,2}$	$U_{3,2}$	$h_{3,3}$	$U_{3,3}$	$h_{3,4}$	$U_{3,4}$	$h_{3,5}$	$U_{3,5}$	$h_{3,6}$	$U_{3,6}$	$h_{3,7}$	$U_{3,7}$
$\theta_{\frac{5}{2}}$	$V_{2,1}$	|	$V_{2,2}$		$V_{2,3}$		$V_{2,4}$		$V_{2,5}$		$V_{2,6}$		$V_{2,7}$	
θ_2	$h_{2,1}$	$U_{2,1}$	$h_{2,2}$	$U_{2,2}$	$h_{2,3}$	$U_{2,3}$	$h_{2,4}$	$U_{2,4}$	$h_{2,5}$	$U_{2,5}$	$h_{2,6}$	$U_{2,6}$	$h_{2,7}$	$U_{2,7}$
$\theta_{\frac{3}{2}}$	$V_{1,1}$	$\llcorner$	$V_{1,2}$	—	$V_{1,3}$	—	$V_{1,4}$	—	$V_{1,5}$	—	$V_{1,6}$	—	$V_{1,7}$	—
θ_1	$h_{1,1}$	$U_{1,1}$	$h_{1,2}$	$U_{1,2}$	$h_{1,3}$	$U_{1,3}$	$h_{1,4}$	$U_{1,4}$	$h_{1,5}$	$U_{1,5}$	$h_{1,6}$	$U_{1,6}$	$h_{1,7}$	$U_{1,7}$

Figure 1
The relative layout of the h, U and V grids.
The dashes mark the boundary of the fluid.
The normal components of velocity vanish across this line.

A fourth grid, vorticity $(\zeta_{i,j})$, can be defined at the points $(\phi_{\frac{2i+1}{2}}, \theta_{\frac{2j+1}{2}})$, lying horizontally between the $V_{i,j}$ points and vertically between the $U_{i,j}$ points.

3.1.1 Infinitesmal vs. finite area elements and line lengths

An infinitesmal differential surface area element on the sphere in our coordinates is $a^2 \cos\theta\, d\theta\, d\phi$. The true area of a finite mesh cell centered at (ϕ, θ) of width $a \cos\theta\, \Delta\phi$, and height $a\, \Delta\theta$ is $2 a^2\, \Delta\phi \sin\left(\frac{\Delta\theta}{2}\right) \cos\theta$. To leading order, the proportional error between the true area and the finite area *element* $a^2 \cos\theta \Delta\phi\, \Delta\theta$ is:

$$\frac{\text{true area - } \textit{finite element}}{\text{true area}} \approx \frac{\Delta\theta^2}{24}. \tag{23}$$

More elaborate finite difference schemes than those set out in the subsequent subsections of this paper that intend to achieve higher accuracy will have to incorporate this second order finite area difference.

Fortunately, finite line elements parallel to the (ϕ, θ) coordinate axes remain analogs of the infinitesmal line elements; $(a \cos\theta\, d\phi,\ a\, d\theta)$ becoming exactly $(a \cos\theta\, \Delta\phi,\ a\, \Delta\theta)$.

3.1.2 Point vs. Area or line averaged values

The variable $h_{i,j}$ is a layer thickness value centered at the point $\phi_i = \phi_0 + i\, \Delta\phi$, $\theta_j = \theta_0 + j\, \Delta\theta$ averaged over a mesh cell of width $a \cos\theta_j \Delta\phi$ and height $a\Delta\theta$. The variable $U_{i,j+\frac{1}{2}}$ is an average of a transport over a line from the point (ϕ_i, θ_j) to the point (ϕ_i, θ_{j+1}). Similarly, $V_{i+\frac{1}{2},j}$ is defined as the average of the transport across the line from (ϕ_i, θ_j) to the point (ϕ_{i+1}, θ_j). In our spherical coordinate system these averages differ from spot values centered on (ϕ_i, θ_j) by terms of second order and higher in $\Delta\theta, \Delta\phi$ times ϕ and θ derivatives of the field.

If we consider in detail the relationship between a spot value of a field $h(\phi_0, \theta_0)$ and the average value of this field over the finite area element of size $2 a^2\, \Delta\phi \sin\left(\frac{\Delta\phi}{2}\right) \cos\theta$ centered at the point (ϕ_0, θ_0) (*called* $\bar{h}$) we find:

$$\bar{h} \approx h(\phi_0, \theta_0) + \frac{(\Delta\phi)^2}{24} \left.\frac{\partial^2 h}{\partial\phi^2}\right|_{\phi_0} + \frac{(\Delta\theta)^2}{24} \left[\left.\frac{\partial^2 h}{\partial\theta^2}\right|_{\theta_0} - 2\tan\theta_0 \left.\frac{\partial h}{\partial\theta}\right|_{\phi_0} \right]$$

$$+ \quad O\left(\Delta\phi^4, \Delta\phi^2\Delta\theta^2, \Delta\theta^4\right) \quad + \quad \ldots \tag{24}$$

A similar second order dependence on the mesh spacing exists between the average value of U between the points (ϕ_i, θ_j) and $(\phi_i,\ \theta_{j+1})$ and its spot value $U(\phi_i, \theta_{j+\frac{1}{2}})$. Likewise for V.

Again these differences between spot and average values must be reflected in more accurate difference formulae.

3.2 Second Order Finite Difference Equations

For convenience in notation, let us follow other authors in ocean modelling [11] and define centered averaging and difference operators:

$$\delta_{mz}\left(W(z)\right) = \left(W\left(z + \frac{m\Delta}{2}\right) - W\left(z - \frac{m\Delta}{2}\right)\right)\bigg/(m\Delta), \tag{25a}$$

$$\overline{W(z)}^{mz} = \left(W\left(z + \frac{m\Delta}{2}\right) + W\left(z - \frac{m\Delta}{2}\right)\right)\bigg/2. \tag{25b}$$

The centered time t applies to all fields unless otherwise noted. Second order accurate finite difference approximations to the equations set out in Section 2.2 take the following forms:

3.2.1 Conservation of Mass

Equation (11):

$$\delta_{2t}h = \frac{-1}{a\cos\theta}\left[\delta_\phi U + \delta_\theta(V\cos\theta)\right] + \text{sources} - \text{sinks}. \tag{26}$$

3.2.2 Velocities

The velocities (u, v) are defined from the transports (U, V) by:

$$u_{i,j} = \frac{U_{i,j}}{\overline{h}^\phi}, \qquad v_{i,j} = \frac{V_{i,j}}{\overline{h}^\theta}. \tag{27}$$

3.2.3 Deformation Tensor Components

Equations (12a) and (12b):

$$e_{\phi\phi} = -e_{\theta\theta} = \delta_\phi\left(\frac{u}{\cos\theta}\right) - \cos\theta\,\delta_\theta\left(\frac{v}{\cos\theta}\right), \tag{28a}$$

$$e_{\phi\theta} = e_{\theta\phi} = \delta_\phi\left(\frac{v}{\cos\theta}\right) + \cos\theta\,\delta_\theta\left(\frac{u}{\cos\theta}\right). \tag{28b}$$

This defines the diagonal elements of the deformation tensor $e_{\phi\phi}$ and $e_{\theta\theta}$ on the $h_{i,j}$ grid and the off-diagonal elements $e_{\phi\theta}$ and $e_{\theta\phi}$ on the $\zeta_{i,j}$ grid. This positions these values exactly where they are needed by the difference operators to calculate the viscous stress contribution to the U and V advection equations.

3.2.4 Velocity transport

Equations (14a) and (14b) are approximated by:

$$\delta_{2t}U \;=\; \frac{-1}{a\cos\theta}\left[\delta_\phi\!\left(\overline{U}^\phi\,\overline{u}^\phi\right) \;+\; \delta_\theta\!\left(\overline{V}^\phi\,\overline{u}^\theta\,\cos\theta\right) \;-\; \sin\theta\,\overline{\left(\overline{v}^\theta\,h\right)}^\phi u \;-\; a\,\Omega\,\sin 2\theta\,\overline{\left(\overline{v}^\theta\,h\right)}^\phi\right]$$

$$-\;\frac{g\,\overline{h}^\phi}{a\cos\theta}\delta_\phi h \;+\; \overline{h}^\phi\,F_\phi \;+\; \frac{\nu}{a^2\cos^2\theta}\left[\delta_\phi(h\cos\theta\,e_{\phi\phi}) \;+\; \delta_\theta\!\left(\overline{h}^{\phi\theta}\cos^2\theta\,e_{\phi\theta}\right)\right]^{t-\Delta t}, \qquad (29\text{a})$$

$$\delta_{2t}V \;=\; \frac{-1}{a\cos\theta}\left[\delta_\phi\!\left(\overline{U}^\theta\,\overline{v}^\phi\right) \;+\; \delta_\theta\!\left(\overline{V}^\theta\,\overline{v}^\theta\,\cos\theta\right) \;+\; \overline{\left(\sin\theta\,\overline{U}\overline{u}^\phi\,h\right)}^\theta \;+\; a\,\Omega\,\overline{\left(\sin 2\theta\,\overline{u}^\phi h\right)}^\theta\right]$$

$$-\;\frac{g\,\overline{h}^\theta}{a}\delta_\theta h \;+\; \overline{h}^\theta\,F_\theta \;+\; \frac{\nu}{a^2\cos^2\theta}\left[\delta_\phi\!\left(\overline{h}^{\phi\theta}\cos\theta\,e_{\theta\phi}\right) \;-\; \delta_\theta\!\left(h\cos^2\theta\,e_{\phi\phi}\right)\right]^{t-\Delta t}. \qquad (29\text{b})$$

3.2.5 Scalar tracers

Scalar fields such as temperature, salinity and tracer concentrations (T, S, C_i) are defined on the $h_{i,j}$ grid. Equation (15) may be approximated in a manner consistent with the previous finite difference equations by:

$$\delta_{2t}(hT) \;=\; \frac{-1}{a\,\cos\theta}\left[\delta_\phi(U\,\overline{T}^\phi) \;+\; \delta_\theta(V\,\overline{T}^\theta\,\cos\theta)\right]$$

$$+\;\frac{\kappa}{a^2\cos^2\theta}\left[\delta_\phi\!\left(\overline{h}^\phi\,\delta_\phi T\right) \;+\; \cos\theta\,\delta_\theta\!\left(\overline{h\cos\theta}^\theta\delta_\theta T\right)\right]^{t-\Delta t}. \qquad (30)$$

3.2.6 Discussion

The horizontal diffusion terms in the three finite difference equations given above are lagged in time one time step for computational stability [16]. This introduces first order timestepping errors. In practice, small values of ν and κ are used, in fact the smallest that allow the code to be stable. Hence a small constant preceeds this first order timestep error and its overall effect is hoped to be small.

In the momentum equations for time stepping U & V, the advection terms have a conservative flux contribution and separate *curvature* terms $(-V\,u\,\tan\theta,\; U\,u\,\tan\theta)$. These *curvature* terms redistribute the momentum, but if the equation $(u \times (14\text{a}) + v \times (14\text{b}))$ is formed to produce a prognostic equation for a kinetic energy $(u\,U + v\,V)$, it is seen that these *curvature* terms make no contribution to the kinetic energy evolution, just as the Coriolis terms also cancel in this combination.

Holland and Lin [5] devised finite difference forms for the Coriolis terms in each equation $\left(-\overline{(\overline{v}^{\theta} f h)}^{\phi}, \overline{(\overline{u}^{\phi} f h)}^{\theta}\right)$ that preserve this conservation of kinetic energy when $\left(\overline{(\overline{v}^{\theta} f h)}^{\phi} u - \overline{(\overline{u}^{\phi} f h)}^{\theta} v\right)$ is summed over a domain where the normal velocity component vanishes at the boundary. This may be confirmed numerically for arbitrary (f, h, U, V) fields satisfying the boundary conditions. When the axis of rotation goes through the coordinate singularities (North & South Poles), this finite difference approximation reduces to the form set out above. The more general forms given in this paragraph should be used when the poles are moved. Other suggested finite difference approximations for the Coriolis terms [18] do not conserve kinetic energy.

This energy conservative finite difference approximation to the Coriolis terms is chosen as a guide to approximating the *curvature* term in each equation.

The scalar tracer finite difference equation preserves locally the property of no change in the scalar field when it is constant, no matter what the velocity field. Any local divergence in the U, V field is reflected only in a change of layer thickness.

3.3 Diagnostic Quantities and Energetics

3.3.1 Vorticity

The normal component of vorticity is defined on a spherical surface grid [4] as:

$$\zeta = \frac{1}{a \cos \theta} \left\{ \frac{\partial v}{\partial \phi} - \frac{\partial (u \cos \theta)}{\partial \theta} \right\}. \tag{31}$$

The finite difference analogue of this equation is:

$$\zeta_{i,j} = \frac{1}{a \cos \theta} \left\{ \delta_{\phi}(v) - \delta_{\theta}(u \cos \theta) \right\}. \tag{32}$$

It may be used as a diagnostic variable.

3.3.2 Kinetic Energy

If the kinetic energy field is defined as $u_{i,j} U_{i,j} + v_{i,j} V_{i,j}$, the two products are defined on separate grids. One or both need to be extrapolated onto a common grid to define a kinetic energy field, although the separate grids may be used for the purposes of integrating kinetic energy over the entire domain. If the kinetic energy is to be combined or compared with the potential energy, then the definition on the $h_{i,j}$ grid would be appropriate, *viz.*:

$$(KE)_{i,j} = \overline{u_{i,j} U_{i,j}}^{\phi} + \overline{v_{i,j} V_{i,j}}^{\theta}. \tag{33}$$

4. TRUNCATION ERROR AND ACCURACY

As the operators (25a,25b) used in constructing the finite difference approximations to the equations of fluid motion are centered, we expect them to represent these differential equations with error terms proportional to even powers of the mesh spacings $(\Delta\phi, \Delta\theta)$ and their products. A full truncation error analysis of the finite difference equations in section 3.2 will reveal this expected behavior.

It is useful to attempt to quantify these error expressions by assuming a surface flow of a simple type and seeing how these finite difference equations reproduce the properties of the analytic equations for this flow. The explicit form taken by the truncation errors in this case may be a useful guide as to the choice of mesh spacing and to the degree the distortion of the mesh near the poles degrades the accuracy of these equations. One such simple flow is that of a rigid shell in unform rotation about some pole centered at the point (ϕ_R, θ_R) .

4.1 Analytic

The surface velocity field (u, v) of a thin spherical shell rotating from west to east about an axis passing through the point (ϕ_R, θ_R) with angular velocity ω is:

$$\vec{u}_R = a\,\omega\,[\cos(\phi - \phi_R)\cos\theta_R\sin\theta - \sin\theta_R\cos\theta, \quad \sin(\phi_R - \phi)\cos\theta_R]. \tag{34}$$

If $\vec{u}_R$ is used to prescribe U and V in the continuity equation (11), it can be shown that the expression in the square brackets vanishes identically. If $\vec{u}_R$ is inserted into the expressions for the viscous stress tensor components then it be shown that they all vanish everywhere, as expected for such a flow.

If these forms are used for U, V, u and v in the bilinear advection terms for the velocity transport time evolution, the terms that arise can be balanced exactly by the pseudo-Coriolis force for an axis of rotation moved to (ϕ_R, θ_R) and a rotation rate of $-\omega$. This is as expected when the apparent forces arising from a rotation of the inertial axes is cancelled by an apparent flow field in the moving frame exactly opposite to the rotation field.

These analytic results will be useful for investigating the accuracy of finite difference representations of the governing equations and alternative formulations.

4.2 Numerical

If the above velocity field is projected onto a grid of size $(\Delta\phi, \Delta\theta)$ and these discrete values are used in the finite difference approximations for the governing equations we can look at the differences between the discrete results and the analytic results in our several cases.

4.2.1 Continuity

The right hand side of equation (26) evaluates to:

$$\cos\theta_R \sin\phi \tan\theta \left[\left(\frac{2}{\Delta\theta}\right) \sin\left(\frac{\Delta\theta}{2}\right) - \left(\frac{2}{\Delta\phi}\right) \sin\left(\frac{\Delta\phi)}{2}\right) \right]. \tag{35}$$

Thus if $\Delta\phi = \Delta\theta$, the terms inside the square bracket cancel and our finite difference scheme is exact for this model flow. The difference between $\Delta\phi$ and $\Delta\theta$ determines the actual truncation error for this scheme and grid. To the lowest order, this error takes the form:

$$\delta_{2t} h \approx \cos\theta_R \sin\phi \tan\theta \left[\frac{(\Delta\phi)^2}{24} - \frac{(\Delta\theta)^2}{24} + O\left((\Delta\phi)^4 - (\Delta\theta)^4\right) + \ldots \right]$$

4.2.2 Advection terms

If the general rigid rotation velocity field (34) is substituted into the advection part of the transport equations, very complicated expressions result. The general level of accuracy of the finite difference approximations here can be illustrated by considering two limits of the rotation velocity field; (i) axis of rotation at the poles and (ii) axis of rotation at the equator.

Let the velocity field for case (i) be $\vec{u} = (-\cos\theta, 0)$. This velocity field in the co-rotating frame exactly recovers the inertial frame and there are no real forces in the advection terms. By inspection we can see that the finite difference approximations to the advection terms in the U transport equation (29a) evaluate to zero for any choice of $\Delta\phi$ and $\Delta\theta$. In the V tranport approximation (29b) the curvature term is cancelled exactly for any mesh by the Coriolis force finite difference approximation.

When we move the axis of rotation to the equator at $\phi_R = 0$, the velocity field takes the form: $\vec{u} = (-\cos\phi \sin\theta, \sin\phi)$. The analytic contribution to the U transport advection is $\sin\phi \cos\phi \cos^2\theta$. The finite difference approximation evaluates to:

$$\sin\phi \cos\phi \cos^2\theta \left[1 - \frac{3 + 14\tan^2\theta}{24} \Delta\phi^2 - \frac{7\cos 2\theta}{24\cos^2\theta} \Delta\theta^2 + \ldots \right] \tag{36}$$

Both of the coefficients of the second order terms are well behaved between $70° N$ and $70° S$, being bounded by 3.3 for the first coefficient and -1.25 for the second in this range. The truncation error properties of the finite difference approximation to the V transport equation (29b) are similar.

4.2.3 Deformation tensor

¿From the components (25a-b) for the deformation tensor, we recover truncation error expressions of the form

$$e_{\phi\phi}(\vec{u}_R) \approx \cos\theta_R \sin\phi \tan\theta \left[\frac{(\Delta\phi)^2}{24} + \left(5 + 6\tan^2(\theta)\right) \frac{(\Delta\theta)^2}{24} + \ldots \right], \qquad (37)$$

$$e_{\phi\theta}(\vec{u}_R) \approx \cos\theta_R \cos\phi \sec\theta \left[\frac{(\Delta\phi)^2}{24} + \left(2 + 6\tan^2(\theta)\right) \frac{(\Delta\theta)^2}{24} + \ldots \right]. \qquad (38)$$

Although the factors involving $6\tan^2\theta$ can get large at high latitudes, these approximations are seen to retain very good second order accuracy over much of the globe.

When these expressions are used in calculating the stress contribution to the velocity transport equations (26a-b), the resultant truncation errors retain their $\left((\Delta\phi)^2, (\Delta\theta)^2\right)$ character and magnitude.

5. ACKNOWLEDGEMENTS

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Appendix A

THE CONNECTION BETWEEN CARTESIAN AND SPHERICAL BASIS VECTORS

It is useful to be able to re-express vectors defined in an absolute $(\vec{i}, \vec{j}, \vec{k})$ Cartesian basis in terms of the local spherical basis ($\vec{e}_\phi$, $\vec{e}_\theta$, $\vec{e}_r$) at the point (ϕ, θ, r) and vice versa. The nine components of the matrix that transforms a vector representation from one basis to the other can be expressed in either the local spherical coordinates (ϕ, θ, r) or in the local cartesian coordinates (x, y, z). We will assume for convenience that the origins of the two coordinate systems coincide and that the two spherical coordinate singularities (the North and South Poles) lie on the z-axis.

A.1 Spherical coordinate representations

To move from a Cartesian basis to a spherical basis we use the relations:

$$\vec{e}_\phi = -\sin\phi \ \vec{i} + \cos\phi \ \vec{j} \tag{A1a}$$

$$\vec{e}_\theta = -\cos\phi\sin\theta \ \vec{i} - \sin\phi\sin\theta \ \vec{j} + \cos\theta \ \vec{k}, \tag{A1b}$$

$$\vec{e}_r = \cos\phi\cos\theta \ \vec{i} + \sin\phi\cos\theta \ \vec{j} + \sin\theta \ \vec{k}. \tag{A1c}$$

To move from a representation in the spherical basis to a representation in the Cartesian basis we use the relations:

$$\vec{i} = -\sin\phi \ \vec{e}_\phi - \cos\phi\sin\theta \ \vec{e}_\theta + \cos\phi\cos\theta \ \vec{e}_r \tag{A2a}$$

$$\vec{j} = \cos\phi \ \vec{e}_\phi - \sin\phi\sin\theta \ \vec{e}_\theta + \sin\phi\cos\theta \ \vec{e}_r \tag{A2b}$$

$$\vec{k} = \cos\theta \ \vec{e}_\theta + \sin\theta \ \vec{e}_r. \tag{A2c}$$

If these transformation rules are expressed as $\vec{v}_s = \vec{\vec{T}}\vec{v}_C$, $\vec{v}_s$ is the 3 components of $\vec{v}$ expressed in the local spherical basis set ($\vec{e}_\phi$, $\vec{e}_\theta$, $\vec{e}_r$), $\vec{v}_C$ is that same vector expressed in the Cartesian basis set $\left(\vec{i}, \vec{j}, \vec{k}\right)$ and $\vec{\vec{T}}$ is a three by three matrix. We note that $\left(\vec{\vec{T}}\right)^{-1} = \left(\vec{\vec{T}}\right)^{t}$. The matrix $\vec{\vec{T}}$ is orthogonal as its transpose is its inverse!

A.2 The Spherical-Cartesian transformations in Cartesian Coordinates

To express $\vec{\vec{T}}$ purely in terms of local Cartesian variables, we define $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, $\rho = (x^2 + y^2)^{\frac{1}{2}}$. Then

$$\vec{\vec{T}} = \begin{pmatrix} -\frac{y}{\rho} & \frac{x}{\rho} & 0 \\[2mm] -\frac{xz}{r\rho} & -\frac{yz}{r\rho} & \frac{\rho}{r} \\[2mm] \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix}. \tag{A3}$$

In this form it is easy to demonstrate that the basis ($\vec{e}_\phi$, $\vec{e}_\theta$, $\vec{e}_r$) defined by (A1 a-c) has the *(required !)* properties:

$$\begin{array}{llllll}
\vec{e}_\phi \cdot \vec{e}_\phi & = & 1, & \vec{e}_\phi \cdot \vec{e}_\theta & = & 0, & \vec{e}_\phi \cdot \vec{e}_r & = & 0, \\[2mm]
\vec{e}_\theta \times \vec{e}_\phi & = & -\vec{e}_r, & \vec{e}_\theta \cdot \vec{e}_\theta & = & 1, & \vec{e}_\theta \cdot \vec{e}_r & = & 0, \\[2mm]
\vec{e}_r \times \vec{e}_\phi & = & \vec{e}_\theta & \vec{e}_r \times \vec{e}_\theta & = & -\vec{e}_\phi, & \vec{e}_r \cdot \vec{e}_r & = & 1.
\end{array} \tag{A4}$$

Appendix B

SPHERICAL SHELL VECTOR CALCULUS

B.1 Definition of Surface Differential Operators on a Sphere

B.1.a Basic Operators

Let Φ be any scalar quantity, and let it be a function of (ϕ, θ) only,

$$(viz. \ \frac{\partial \Phi}{\partial r} \equiv 0).$$

Let $\vec{F}$ be a vector quantity such as velocity $\vec{v}$, or mass transport $\vec{V}$. $\vec{F}$ is assumed to lie in the local tangent plane and to describe a field everywhere nearly parallel to the spherical layer it is embedded in. Quantities advected by $\vec{v}$ (or $\vec{V}$) are assumed to maintain vertical coherence across a fluid layer (c. f. [14] p 63). That is, two particles lying on the same radial vector at the top and bottom of a fluid layer respectively remain on a single radial vector as they are advected by the velocity field of that layer. This implies that $\vec{u}$ is obliged to assume the form:

$$\vec{u}(\phi, \theta, r) \equiv \left(\frac{r}{a} u'(\phi, \theta), \ \frac{r}{a} v'(\phi, \theta), \ w(\phi, \theta, r) \right), \tag{B1a}$$

where $w(\phi, \theta, r)$ takes the form $\frac{Ar^3 + B}{r^2}$ necessary to maintain zero divergence for the velocity and to match the motions of the upper and lower boundaries of the layer. in what follows, let us consider a general vector $\vec{F}$ is expressed in coordinate form as:

$$\vec{F} \equiv \left(\frac{r}{a} F'_\phi(\phi, \theta), \ \frac{r}{a} F'_\theta(\phi, \theta), F_r(\phi, \theta, r) \right). \tag{B1b}$$

In many ocean model contexts, the radial component of the velocity vector is ignored. For these purposes it is useful to consider just the horizontal' components of velocity. Let us define $\vec{u}_2$ to be:

$$\vec{u}_2(\phi, \theta, r) \equiv \left(\frac{r}{a} \right) \left(u'(\phi, \theta), v'(\phi, \theta), 0 \right), \tag{B2}$$

and similarly $\vec{F}_2$ to be any locally horizontal vector field.

The standard invariant operators of vector calculus are now set out in explicit coordinate form using our spherical surface variables (ϕ, θ) and assuming a functional form for vectors defined as in (B1b) or (B2) and for scalars dependent upon (ϕ, θ) only.[1]

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[1] c.f. Batchelor (1970), pp 598-601 or Gill (1982), pp 92-93

B.1.b   Gradient $(\nabla_2 \Phi)$

$$\nabla_2 \Phi \;=\; \frac{1}{r \cos\theta} \frac{\partial \Phi}{\partial \phi} \, \vec{e}_\phi \;+\; \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \, \vec{e}_\theta. \tag{B3}$$

This operator retains an r dependence throughout a spherical shell and will have to be vertically averaged with care.

B.1.c   Divergence $(\nabla \cdot \vec{F})$

$$\nabla \cdot \vec{F} \;=\; \frac{1}{a \cos\theta} \frac{\partial F'_\phi}{\partial \phi} \;+\; \frac{1}{a \cos\theta} \frac{\partial (F'_\theta \cos\theta)}{\partial \theta} \;+\; \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r}. \tag{B4}$$

The first two terms are independent of r. This expression may be used to define an r component to a vector field to make it divergence free. We will define $\nabla_2 \cdot$ to be:

$$\nabla_2 \cdot \vec{F}_2 \;=\; \frac{1}{a \cos\theta} \frac{\partial F'_\phi}{\partial \phi} \;+\; \frac{1}{a \cos\theta} \frac{\partial (F'_\theta \cos\theta)}{\partial \theta}. \tag{B5}$$

B.1.d   Curl $(\nabla \times \vec{F})$

$$\nabla \times \vec{F} \;=\; \left(\frac{1}{r} \frac{\partial F_r}{\partial \theta} - \frac{2 F'_\theta}{a} \right) \vec{e}_\phi \;+\; \left(\frac{2 F'_\phi}{a} - \frac{1}{r \cos\theta} \frac{\partial F_r}{\partial \phi} \right) \vec{e}_\theta$$

$$+\; \frac{1}{a \cos\theta} \left\{ \frac{\partial F'_\theta}{\partial \phi} - \frac{\partial (F'_\phi \cos\theta)}{\partial \theta} \right\} \vec{e}_r \tag{B6}$$

When $\vec{F}_2$ is the velocity field, the $\vec{e}_r$ is the normal component of the spherical surface flow vorticity, commonly labeled ζ. [2]. Note that there are both $\vec{e}_\phi$ and $\vec{e}_\theta$ components of the full vorticity vector for all non-zero horizontal velocity fields, even in the absence of any vertical velocity.

B.1.e   Scalar Laplacian $(\nabla_2^2 \Phi)$

$$\nabla_2^2 \Phi \;=\; \frac{1}{r^2 \cos^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} \;+\; \frac{1}{r^2 \cos\theta} \frac{\partial}{\partial \theta} \left(\frac{\partial \Phi}{\partial \theta} \cos\theta \right). \tag{B7}$$

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[2]Pedlosky (1987), p. 55, eqn. (2.9.20) is incorrect. Consider Stokes's theorem :
   $\int_S (\nabla \times \vec{v}) \cdot d\vec{s} = \oint_C \vec{v} \cdot d\vec{\ell}$ for the infinitesmal surface area element $\Delta\phi \cos\theta \, \Delta\theta$.

B.1.f   **Vector Laplacian** $(\nabla^2 \vec{F}_2)$

$$\nabla^2 \vec{F} = \frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 F'_\phi}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial F'_\phi}{\partial\theta}\right) + \cos 2\theta F'_\phi - 2\sin\theta\frac{\partial F'_\theta}{\partial\phi}\right]\vec{e}_\phi$$

$$+ \frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 F'_\theta}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial F'_\theta}{\partial\theta}\right) + \cos 2\theta F'_\theta + 2\sin\theta\frac{\partial F'_\phi}{\partial\phi}\right]\vec{e}_\theta$$

$$- \frac{2}{ra\cos\theta}\left[\frac{\partial F'_\phi}{\partial\phi} + \frac{\partial(F'_\theta\cos\theta)}{\partial\theta}\right]\vec{e}_r. \tag{B8}$$

We recognize the $\vec{e}_r$ component as $-\frac{2}{r}\nabla_2\cdot\vec{F}_2$.

B.2   **Higher Vector Derivatives of the velocity field**

B.2.a   **Momentum Advection**

$$(\vec{u}_2\cdot\nabla)\vec{u}_2 = \frac{r}{a}\left[\frac{u'}{a\cos\theta}\frac{\partial u'}{\partial\phi} + \frac{v'}{a}\frac{\partial u'}{\partial\theta} - \frac{\tan\theta}{a}u'v'\right]\vec{e}_\phi$$

$$+ \frac{r}{a}\left[\frac{u'}{a\cos\theta}\frac{\partial v'}{\partial\phi} + \frac{v'}{a}\frac{\partial v'}{\partial\theta} + \frac{\tan\theta}{a}u'u'\right]\vec{e}_\theta \tag{B9}$$

$$- \frac{r}{a}\left[\frac{u'u' + v'v'}{a}\right]\vec{e}_r$$

B.2.b   **curl curl**

$$\nabla\times(\nabla\times\vec{u}_2) = -\frac{1}{ra\cos^2\theta}\left[\cos\theta\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u'}{\partial\theta}\right) + \cos 2\theta\,u' - \sin\theta\frac{\partial v'}{\partial\phi} - \cos\theta\frac{\partial^2 v'}{\partial\phi\partial\theta}\right]\vec{e}_\phi$$

$$- \frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 v'}{\partial\phi^2} + 2\cos^2\theta\,v' - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u'}{\partial\phi}\right)\right]\vec{e}_\theta$$

$$+ \frac{2}{ra\cos\theta}\left[\frac{\partial u'}{\partial\phi} + \frac{\partial(v'\cos\theta)}{\partial\theta}\right]\vec{e}_r. \tag{B10}$$

Note: $\nabla\times(\nabla\times\vec{u}_2) \neq -\nabla_2^2\vec{u}_2$ because $\nabla_2\cdot\vec{u}_2 \neq 0$.

B.2.c   **grad div**

$$\nabla(\nabla\cdot\vec{u}_2) = +\frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 u'}{\partial\phi^2} + \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial v'}{\partial\phi}\right)\right]\vec{e}_\phi$$

$$+ \frac{1}{ra}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\cos\theta}\left\{\frac{\partial u'}{\partial\phi} + \frac{\partial(v'\cos\theta)}{\partial\theta}\right\}\right)\right]\vec{e}_\theta$$

$$+ 0\cdot\vec{e}_r \tag{B11}$$

B.2.d   div grad

$$(\nabla \cdot \nabla)\, \vec{u}_2 \;=\; + \frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 u'}{\partial\phi^2} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\cos\theta\,\frac{\partial u'}{\partial\theta}\right)\right]\vec{e}_\phi$$

$$+ \frac{1}{ra\cos^2\theta}\left[\frac{\partial^2 v'}{\partial\phi^2} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\cos\theta\,\frac{\partial v'}{\partial\theta}\right)\right]\vec{e}_\theta$$

$$+\, 0 \cdot \vec{e}_r \tag{B12}$$

B.3   The rate-of-strain tensor $\left(\vec{\vec{\mathcal{E}}}\right)$

$$\vec{\vec{\mathcal{E}}} \;=\; \begin{pmatrix} \mathcal{E}_{\phi\phi} & \mathcal{E}_{\phi\theta} & \mathcal{E}_{\phi r} \\ \mathcal{E}_{\theta\phi} & \mathcal{E}_{\theta\theta} & \mathcal{E}_{\theta r} \\ \mathcal{E}_{r\phi} & \mathcal{E}_{r\theta} & \mathcal{E}_{rr} \end{pmatrix}. \tag{B13}$$

Assuming w is not zero:

$$\mathcal{E}_{rr} \;=\; \frac{\partial w}{\partial r}; \tag{B14}$$

$$\mathcal{E}_{\theta r} \;=\; \mathcal{E}_{r\theta} \;=\; \frac{1}{2r}\frac{\partial w}{\partial\theta}; \tag{B15}$$

$$\mathcal{E}_{\phi r} \;=\; \mathcal{E}_{r\phi} \;=\; \frac{1}{2r\cos\theta}\frac{\partial w}{\partial\phi}; \tag{B16}$$

Assuming w is zero:

$$\mathcal{E}_{\theta\theta} \;=\; \frac{1}{a}\frac{\partial v'}{\partial\theta}; \tag{B17}$$

$$\mathcal{E}_{\phi\phi} \;=\; \frac{1}{a\cos\theta}\frac{\partial u'}{\partial\phi} - \frac{v'\tan\theta}{a}; \tag{B18}$$

$$\mathcal{E}_{\phi\theta} \;=\; \mathcal{E}_{\theta\phi} \;=\; \frac{\cos\theta}{2a}\frac{\partial}{\partial\theta}\left(\frac{u'}{\cos\theta}\right) + \frac{1}{2a\cos\theta}\frac{\partial v'}{\partial\phi}. \tag{B19}$$